Koiter's Buckling Mode for Short Cylindrical Shells

C. P. MANGELSDORF*

University of Pittsburgh, Pittsburgh, Pa.

Theme

WITH the introduction of a unique buckling mode in which the primary manifestation of displacement is in the axial, rather than the radial direction, Koiter[1] has shown by means of an upper bound that for very short shells with modified simple supports, the critical stress must be much smaller even than one-half of the classical value. The purpose of the present work is to determine specific values of the critical stress in Koiter's mode for the cases of axial and uniform pressure loading.

Contents

Koiter's buckling mode for circular shells can be visualized by developing the buckled shell surface (the surface is still developable since the radial displacements are considered negligible). The extremities of this developed surface, the end circumferences, form parallel sinusoidal curves. This mode is by definition an unsymmetrical mode with respect to the central cross section.

Procedure

All working equations (except the assumed deflections) are derived from operations performed on the energy functional, that is, the summation of all the quadratic energy terms associated with the deformation during buckling, as developed by Koiter. This is given in Ref. 1 and has been modified in Ref. 2 to a more suitable form by the addition and subtraction of terms which are shown to be of negligible magnitude.

The equilibrium equations in the x, y, and z directions are achieved by applying variational operations to the modified energy functional (including appropriate load terms) with respect to u, v, and w, the shell displacements and their derivatives in the longitudinal and circumferential directions in turn, and setting equal to zero.

Two of the four boundary conditions are those traditionally associated with simple supports, namely, that the radial deflections and the axial bending moments at the ends of the cylinder are zero. For the other two conditions it is assumed that the circumferential shear stress τ_{xy} and the change in axial stress σ_x are both zero at the ends. The last three conditions are so-called "natural" boundary conditions derivable from the modified energy functional.

Assumed deflections are operated upon according to the equilibrium equations and these, in turn, are reduced to a single characteristic equation in terms of the unknown constant coefficients of the radial deflection w. The roots of this equation, which are functions of the shell parameters, the loading, and the wave numbers in the axial and circumferential directions, are then determined. The assumed deflections are used in the boundary equations at both ends of the shell to obtain eight homogeneous linear equations involving the roots of the characteristic equation and eight constants of integration. These equations are separated into two sets of four, corresponding to symmetrical and antisymmetrical modes. Each set forms a determinant which is equated to zero and the buckling load is thereby obtained. The actual evaluation of the buckling load from the determinant involves the solution of a complicated transcendental equation which was simplified by the use of small angle approximations, where applicable, and solved by trial and error. A computer was employed in the latter process.

Axially Loaded Case

In Fig. 1 is shown the results of the solution.† The load is expressed in terms of the dimensionless parameter $\lambda = \sigma_x/\sigma_{cl}$, where $\sigma_{cl} = Eh/[3(1-\nu^2)]^{1/2}R$ is the classical buckling load. The shell parameters are expressed by the ratio $L/(Rh)^{1/2}$, where L is the length, R the radius, and h the thickness. The solution for antisymmetrical buckling has been evaluated for the circumferential wave numbers $q = 1$ and 2 while the solution for symmetrical modes is plotted for small values of q and for values of R/h within the range of 30–300. Also included in Fig. 1 are the average results of Hoff and Soong[4] for symmetrical and antisymmetrical modes.

Of primary interest are the antisymmetrical curves where it will be seen that λ, for values of $L/(Rh)^{1/2}$ less than one, is given approximately by the expression

$$\lambda = [(1-\nu^2)/3]^{1/2}(L^2/Rh)q^2$$

This relation yields values of λ less than the upper bound proposed in Ref. 1 by only a factor of $1-\nu^2$. It appears that the buckling load approaches zero as the length of the shell diminishes. Such a consequence is to be expected since the resistance to buckling in Koiter's mode is derived primarily from bending of a cross section for which the depth is L and the width is h.

Although the solution for antisymmetrical modes yields a relation between λ and $L/(Rh)^{1/2}$ which is independent of the ratio h/R, the use of the same parameters to display symmetrical modes introduces a dependence upon h^2/R^2. However, actual evaluation reveals that variations of h/R between 1/30 and

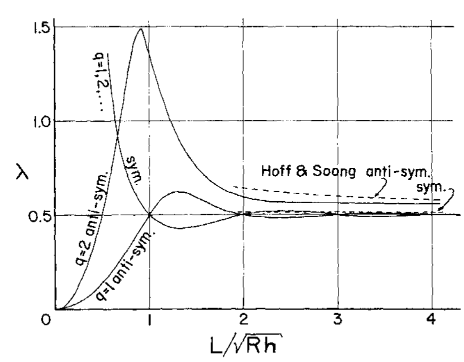

Fig. 1 Relative buckling stresses for axially loaded cylinders $\lambda = \sigma_x/\sigma_{cl}$.

† Identical results for the axially loaded case have been achieved by Simmonds and Danielson[3] using similar equations.

Received October 2, 1970; synoptic received June 22, 1971; revision received August 16, 1971. Full paper (author's Rept. No. J5281) available from National Technical Information Service, Springfield, Va. 22151, as N71-75097, at the standard price (available upon request). The work synopsized herein was accomplished in the academic year 1966–1967 during which the author was employed by the Technische Hogeschool in Delft, The Netherlands. The author is grateful to W. T. Koiter for the opportunity to have worked on this project, for his patient supervision, and for his permission to publish the results.

Index category: Structural Stability Analysis.

* Associate Professor of Civil Engineering.

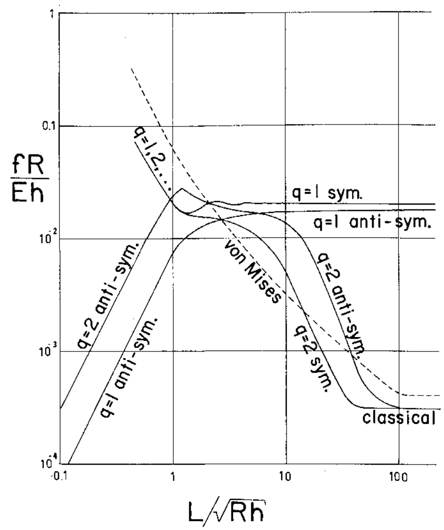

Fig. 2 Buckling pressures for uniformly loaded cylinders $h/R = 1/30$.

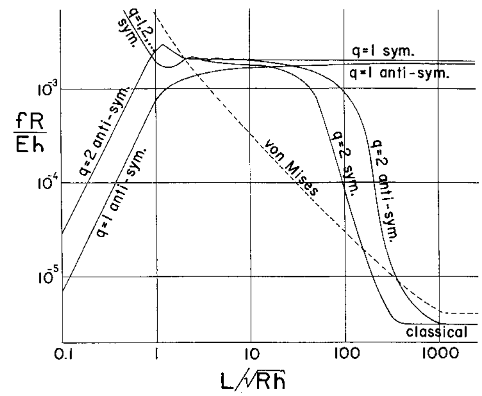

Fig. 3 Buckling pressures for uniformly loaded cylinders $h/R = 1/300$.

1/300, and of q between 2 and 5 have insignificant influence when plotted to the scale of Fig. 1.

Uniform Pressure Case

The results of the mathematical manipulations for the pressure case are shown in Fig. 2 and 3 which are evaluations for two different radius-to-thickness ratios, R/h, of 30 and 300, respectively. Here the load is expressed as $f_{cr}R/hE$, where f_{cr} is the buckling pressure. In addition to symmetric and anti-symmetric modes for $q = 1$ and 2, a plot of the classical solution for infinitely long shells ($f_{cr} = Eh^3/4(1-\nu^2)R^3$ for $q = 2$) and of the von Mises[5] solution including shorter cylinders are shown for comparison. In the latter instance only an approximate curve is given, the exact solution being a series of intersecting loops for various values of the circumferential wave numbers.

Again, as in the axially loaded case, the significant results are for very short cylinders buckling in Koiter's antisymmetric mode. For $L/(Rh)^{1/2} < 1$ the curves of Fig. 2 and 3 predict critical axial stresses of exactly one-third those in Fig. 1 indicating that the circumferential stresses fR/h are as effective in the occurrence of buckling as are the axial ones. Indeed, it can even be suggested that any combination of axial and radial loadings which produces a sum of compressive membrane stresses equal to the axial value given in Fig. 1 will produce buckling.

On the other hand, the curves in Fig. 2 and 3 for the symmetrical modes, $q = 1$, are numerically identical, for very short cylinders, to the same curve in Fig. 1 suggesting that for this manner of buckling, axial stresses play the most important role.

References

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